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Dual Lie Bialgebra Structures of Poisson Types

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Abstract Let $\mathcal{A} = \mathbb{F}[x, y]$ be the polynomial algebra on two variables x, y over an algebraically closed field $\mathbb{F}$ of characteristic zero. Under the Poisson bracket, $\mathcal{A}$ is equipped with a natural Lie algebra structure. It is proven that the maximal good subspace of $\mathcal{A}^*$ induced from the multiplication of the associative commutative algebra $\mathcal{A}$ coincides with the maximal good subspace of $\mathcal{A}^*$ induced from the Poisson bracket of the Poisson Lie algebra $\mathcal{A}$. Based on this, structures of dual Lie bialgebras of the Poisson type are investigated. As by-products, five classes of new infinite dimensional Lie algebras are obtained.

Keywords Poisson algebra, Virasoro-like algebra, Lie bialgebra, dual Lie bialgebra, good subspace

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1 Introduction

Lie bialgebras, having close relations with Yang-Baxter equations [6], are important ingredients in quantum groups, which have drawn more and more attentions in literature (e.g., [2, 3, 5, 8–17, 26, 28, 29]). Michaelis [10] investigated structures of Witt type Lie bialgebras. Ng and Taft [16] gave a classification of this type Lie bialgebras, and obtained that all structures of Lie bialgebras on the one sided Witt algebra, the Witt algebra and the Virasoro algebra are coboundary triangular (cf. [15]). For the cases of generalized Witt type Lie algebras and generalized Virasoro-like Lie algebras, the authors of [17, 28] proved that all structures of Lie bialgebras on them are coboundary triangular. Similar results hold for some other kinds of Lie algebras (cf., e.g., [28, 29]).

From the examples of infinite dimensional Lie bialgebras constructed in [10], many infinite dimensional Lie bialgebra structures we know are coboundary triangular. It may sound that coboundary triangular Lie bialgebras are relatively simple. However, they are not trivial in the sense that many natural problems associated with them remain open (see, also Remark 4.7). For example, even for the (two-sided) Witt algebra and the Virasoro algebra, a complete classification of coboundary triangular Lie bialgebra structures on them is still an open problem. Nevertheless, not much on representations of infinite dimensional Lie bialgebras is known. From the viewpoint of Lie bialgebras, considering dual Lie bialgebra structures may help us understand more on infinite dimensional Lie bialgebra structures. For instance, by considering structures of dual Lie bialgebras of Witt and Virasoro types, the authors of [18] surprisingly

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obtained some new series of infinite dimensional Lie algebras. In the present paper, we study structures of dual Lie bialgebras of Poisson type. One may have noticed that the dual of a finite dimensional Lie bialgebra is naturally a Lie bialgebra, and so one would not predict anything new in this case. However, for the cases of infinite dimensional Lie bialgebras, the situations become quite different, which can be seen in the following contents.

Let us recall the definition of Poisson algebras here: a *Poisson algebra* is a triple $(\mathcal{P},[\cdot,\cdot],\cdot)$ such that $(\mathcal{P},[\cdot,\cdot])$ is a Lie algebra, $(\mathcal{P},\cdot)$ is an associative algebra, and the following *Leibniz rule* holds:

$$[a,bc]=[a,b]c+b[a,c] \quad \text{for} \quad a,b,c\in\mathcal{P}. \tag{1.1}$$

In particular, for any commutative associative algebra $(\mathcal{A},\cdot)$, and any commutative derivations ∂_1,∂_2 of $\mathcal{A}$, we obtain a *Poisson algebra* $(\mathcal{A},[\cdot,\cdot],\cdot)$ with Lie bracket $[\cdot,\cdot]$ defined as follows.

$$[a,b]=\partial_1(a)\partial_2(b)-\partial_2(a)\partial_1(b) \quad \text{for} \quad a,b\in\mathcal{A}. \tag{1.2}$$

If we take $\mathcal{A}=\mathbb{F}[x^{\pm1},y^{\pm1}]$ (where $\mathbb{F}$ is an algebraically closed field of characteristic zero) and $\partial_1=x\frac{\partial}{\partial x}$, $\partial_2=y\frac{\partial}{\partial y}$, then we obtain the *Virasoro-like algebra* $(\mathcal{A},[\cdot,\cdot])$ with basis $\{x^iy^j\,|\,i,j\in\mathbb{Z}\}$ and Lie bracket

$$[x^iy^j,x^ky^\ell]=(i\ell-jk)x^{i+k}y^{j+\ell} \quad \text{for} \quad i,j,k,\ell\in\mathbb{Z}. \tag{1.3}$$

The Virasoro-like algebra (1.3) can be generalized as follows: For any *nondegenerate* additive subgroup Γ of $\mathbb{F}^2$ (i.e., Γ contains an $\mathbb{F}$-basis of $\mathbb{F}^2$), we have the group algebra $\mathcal{A}=\mathbb{F}[\Gamma]$ with basis $\{L_\alpha\,|\,\alpha\in\Gamma\}$ and multiplication defined by $\mu(L_\alpha,L_\beta)=L_{\alpha+\beta}$ for $\alpha,\beta\in\Gamma$. Then we have the (generalized) Virasoro-like algebra $(\mathcal{A},\varphi)$ with Lie bracket φ defined by

$$\varphi(L_\alpha,L_\beta)=(\alpha_1\beta_2-\beta_1\alpha_2)L_{\alpha+\beta} \quad \text{for} \quad \alpha=(\alpha_1,\alpha_2),\ \beta=(\beta_1,\beta_2)\in\Gamma. \tag{1.4}$$

Furthermore, if we take $\mathcal{A}=\mathbb{F}[x,y]$ and $\partial_1=\frac{\partial}{\partial x}$, $\partial_2=\frac{\partial}{\partial y}$, then we obtain the *classical Poisson algebra* $(\mathbb{F}[x,y],[\cdot,\cdot],\cdot)$, whose Lie bracket is given by

$$[f,g]=J(f,g) \quad \text{for} \quad f,g\in\mathbb{F}[x,y], \tag{1.5}$$

where $J(f,g):=\begin{vmatrix}\frac{\partial f}{\partial x} & \frac{\partial f}{\partial y}\\ \frac{\partial g}{\partial x} & \frac{\partial g}{\partial y}\end{vmatrix}$ is the *Jacobian determinant* of f and g.

The reason we have a special interest in the classical Poisson algebra also lies in the fact that this algebra is closely related to the distinguished Jacobian conjecture (e.g., [30, 31]), which can be stated as "any non-zero endomorphism of $(\mathbb{F}[x,y],[\cdot,\cdot],\cdot)$ is an isomorphism". One observes that a Jacobi pair (f,g) (i.e., $f,g\in\mathbb{F}[x,y]$ satisfying $J(f,g)\in\mathbb{F}\backslash\{0\}$) corresponds to a solution $r=f\otimes fg-fg\otimes f$ of the classical Yang-Baxter Equation (cf. (2.1)), thus gives rise to a Lie bialgebra structure on $\mathbb{F}[x,y]$.

The paper is organized as follows. Some definitions and preliminary results are briefly recalled in Section 2. Then in Section 3, structures of dual coalgebras of $\mathbb{F}[x,y]$ are addressed. Finally in Section 4, structures of dual Lie bialgebras of Poisson type are investigated. The main results of the present paper are summarized in Theorems 3.2, 4.4, 4.6, 4.8 and 4.9.

2 Definitions and preliminary results

Throughout the paper, all vector spaces are assumed to be over an algebraically closed field $\mathbb{F}$ of characteristic zero. As usual, we use $\mathbb{Z}_+$ to denote the set of nonnegative integers. We briefly recall some notions on Lie bialgebras, for details, we refer readers to, e.g., [6, 17].

Definition 2.1. 1. A *Lie bialgebra* is a triple $(L,[\cdot,\cdot],\delta)$ such that $(L,[\cdot,\cdot])$ is a Lie algebra, (L,δ) is a Lie coalgebra, and $\delta:L\to L\otimes L$ is a derivation, namely, $\delta[x,y]=x\cdot\delta(y)-y\cdot\delta(x)$ for $x,y\in L$, where $x\cdot(y\otimes z)=[x,y]\otimes z+y\otimes[x,z]$ for $x,y,z\in L$.

2. A Lie bialgebra $(L,[\cdot,\cdot],\delta)$ is *coboundary* if δ is coboundary in the sense that there exists $r\in L\otimes L$ written as $r=\sum r^{[1]}\otimes r^{[2]}$, such that $\delta(x)=x\cdot r$ for $x\in L$.

3. A coboundary Lie bialgebra $(L, [\cdot,\cdot], \delta)$ is *triangular* if r satisfies the following *classical Yang-Baxter Equation* (CYBE),
$$C(r) = [r_{12}, r_{13}] + [r_{12}, r_{23}] + [r_{13}, r_{23}] = 0, \tag{2.1}$$
where $r_{12} = \sum r^{[1]} \otimes r^{[2]} \otimes 1$, $r_{13} = \sum r^{[1]} \otimes 1 \otimes r^{[2]}$, $r_{23} = \sum r^{[1]} \otimes 1 \otimes r^{[2]}$ are elements in $U(L) \otimes U(L) \otimes U(L)$, and $U(L)$ is the universal enveloping algebra of L.

Two Lie bialgebras $(\mathfrak{g}, [\cdot,\cdot], \delta)$ and $(\mathfrak{g}', [\cdot,\cdot]', \delta')$ are said to be *dually paired* if their bialgebra structures are related via

$$\langle [f,h]', \xi \rangle = \langle f \otimes h, \delta\xi \rangle, \quad \langle \delta' f, \xi \otimes \eta \rangle = \langle f, [\xi, \eta] \rangle \ \text{ for } f, h \in \mathfrak{g}',\ \xi, \eta \in \mathfrak{g}, \tag{2.2}$$

where $\langle \cdot,\cdot \rangle$ is a nondegenerate bilinear form on $\mathfrak{g}' \times \mathfrak{g}$, which is naturally extended to a nondegenerate bilinear form on $(\mathfrak{g}' \otimes \mathfrak{g}') \times (\mathfrak{g} \otimes \mathfrak{g})$. In particular, if $\mathfrak{g}' = \mathfrak{g}$ as a vector space, then $\mathfrak{g}$ is called a *self-dual Lie bialgebra*.

The following result whose proof is straightforward can be found in [9].

Proposition 2.2. Let $(\mathfrak{g}, [\cdot,\cdot], \delta)$ be a finite dimensional Lie bialgebra, then so is the linear dual space $\mathfrak{g}^* := \mathrm{Hom}_{\mathbb{F}}(\mathfrak{g}, \mathbb{F})$ by dualisation, namely $(\mathfrak{g}^*, [\cdot,\cdot]', \delta')$ is the Lie bialgebra defined by (2.2) with $\mathfrak{g}' = \mathfrak{g}^*$. In particular, $\mathfrak{g}$ and $\mathfrak{g}^*$ are dually paired.

Thus a finite dimensional Lie biallgebra $(\mathfrak{g}, [\cdot,\cdot], \delta)$ is always self-dual as there exists a vector space isomorphism $\mathfrak{g} \to \mathfrak{g}^*$ which pulls back the bialgebra structure on $\mathfrak{g}^*$ to $\mathfrak{g}$ to obtain another bialgebra structure on $\mathfrak{g}$ to make it to be self-dual. However, in sharp contrast to the finite dimensional case, infinite dimensional Lie bialgebras are not self-dual in general.

For convenience, we denote by φ the Lie bracket of Lie algebra $(\mathfrak{g}, [\cdot,\cdot])$, which can be regarded as a linear map $\varphi : \mathfrak{g} \otimes \mathfrak{g} \to \mathfrak{g}$. Let $\varphi^* : \mathfrak{g}^* \to (\mathfrak{g} \otimes \mathfrak{g})^*$ be the dual of φ.

Definition 2.3. [11] Let $(\mathfrak{g}, \varphi)$ be a Lie algebra over $\mathbb{F}$. A subspace V of $\mathfrak{g}^*$ is called a *good subspace* if $\varphi^*(V) \subset V \otimes V$. Denote $\Re = \{V \mid V$ is a good subspace of $\mathfrak{g}^*\}$. Then $\mathfrak{g}^\circ = \sum_{V \in \Re} V$, is also a good subspace of $\mathfrak{g}^*$, which is obviously the maximal good subspace of $\mathfrak{g}^*$.

It is clear that if $\mathfrak{g}$ is a finite dimensional Lie algebra, then $\mathfrak{g}^\circ = \mathfrak{g}^*$.

Proposition 2.4. [11] For any good subspace V of $\mathfrak{g}^*$, the pair (V, φ^*) is a Lie coalgebra. In particular, $(\mathfrak{g}^\circ, \varphi^*)$ is a Lie coalgebra.

For any Lie algebra $\mathfrak{g}$, the dual space $\mathfrak{g}^*$ has a natural right $\mathfrak{g}$-module structure defined for $f \in \mathfrak{g}^*$ and $x \in \mathfrak{g}$ by
$$(f \cdot x)(y) = f([x, y]) \ \text{ for } \ y \in \mathfrak{g}.$$
We denote $f \cdot \mathfrak{g} = \mathrm{span}\{f \cdot x \mid x \in \mathfrak{g}\}$, the *space of translates* of f by elements of $\mathfrak{g}$.

We summarize some results of [2, 3, 5, 8] as follows.

Proposition 2.5. Let $\mathfrak{g}$ be a Lie algebra. Then

1. $\mathfrak{g}^\circ = \{f \in \mathfrak{g}^* \mid f \cdot \mathfrak{g}$ is finite dimensional$\}$.
2. $\mathfrak{g}^\circ = (\varphi^*)^{-1}(\mathfrak{g}^* \otimes \mathfrak{g}^*)$, the preimage of $\mathfrak{g}^* \otimes \mathfrak{g}^*$ in $\mathfrak{g}^*$.

The notion of good subspaces of an associative algebra can be defined analogously. In the next two sections, we shall investigate $\mathfrak{g}^\circ$ for some associative or Lie algebras $\mathfrak{g}$.

3 The structure of $\mathbb{F}[x, y]^\circ$

Let $(\mathcal{A}, \mu, \eta)$ be an associative $\mathbb{F}$-algebra with unit, where μ and η are respectively the multiplication $\mu : \mathcal{A} \otimes \mathcal{A} \to \mathcal{A}$ and the unit $\eta : \mathbb{F} \to \mathcal{A}$, satisfying

$$\mu \circ (id \otimes \mu) = \mu \circ (\mu \otimes id)) : \qquad \mathcal{A} \otimes \mathcal{A} \otimes \mathcal{A} \to \mathcal{A},$$
$$(\eta \otimes id)(k \otimes a) = (id \otimes \eta)(a \otimes k) : \quad \mathbb{F} \otimes \mathcal{A} \cong \mathcal{A} \otimes \mathbb{F} \cong \mathcal{A},$$

for $k \in \mathbb{F}$, $a \in \mathcal{A}$. Then a coassociative coalgebra is a triple (C, Δ, η), which is obtained by conversing arrows in the definition of an associative algebra. Namely, $\Delta : C \to C \otimes C$ and $\eta : \mathbb{F} \to C$ are respectively comultiplication and counit of C, satisfying

$$\begin{aligned}
&(\Delta \otimes id) \circ \Delta = (id \otimes \Delta) \circ \Delta : \quad C \to C \otimes C \otimes C,\\
&(\eta \otimes id) \circ \Delta = (id \otimes \eta) \circ \Delta : \quad C \to C \otimes C \to \mathbb{F} \otimes C \cong C \otimes \mathbb{F} \cong C.
\end{aligned}$$

For any vector space $\mathcal{A}$, there exists a natural injection $\rho : \mathcal{A}^* \otimes \mathcal{A}^* \to (\mathcal{A} \otimes \mathcal{A})^*$ defined by $\rho(f, g)(a, b) = \langle f, a\rangle\langle g, b\rangle$ for $f, g \in \mathcal{A}^*$ and $a, b, \in \mathcal{A}$. In case $\mathcal{A}$ is finite dimensional, ρ is an isomorphism. If $(\mathcal{A}, \mu)$ is associative, the multiplication μ induces the map $\mu^* : \mathcal{A}^* \to (\mathcal{A} \otimes \mathcal{A})^*$. If $\mathcal{A}$ is finite dimensional, then the isomorphism ρ insures that $(\mathcal{A}^*, \mu^*, \eta^*)$ is a coalgebra, where for simplicity, μ^* denotes the composition of the maps: $\mathcal{A}^* \xrightarrow{\mu^*} (\mathcal{A} \otimes \mathcal{A})^* \xrightarrow{(\rho)^{-1}} \mathcal{A}^* \otimes \mathcal{A}^*$.

Now let $(\mathcal{A}, \mu)$ be a commutative associative algebra. Then $\mathcal{A}^\circ = (\mu^*)^{-1}(\mathcal{A}^* \otimes \mathcal{A}^*)$ (cf. [26] and Proposition 2.5). For $\partial \in \mathrm{Der}(\mathcal{A})$ and $f \in \mathcal{A}^\circ$, using

$$\partial\mu = \mu(id \otimes \partial + \partial \otimes id), \quad \mu^*\partial^*(f) = (id \otimes \partial^* + \partial^* \otimes id)\mu^*(f) \in \mathcal{A}^* \otimes \mathcal{A}^*,$$

we obtain $\partial^*(\mathcal{A}^\circ) \subset \mathcal{A}^\circ$. Thus, we observe that there are two natural approaches to produce Lie coalgebras from some subspaces of $\mathcal{A}^*$. One is induced from the associative structure of $\mathcal{A}$ as follows: First we have the cocommutative coassociative coalgebra $(\mathcal{A}^\circ, \mu^\circ)$ with $\mu^\circ := \mu^*|_{\mathcal{A}^\circ}$. Then we obtion the Lie coalgebra $\mathcal{A}^\circ_\mu := (\mathcal{A}^\circ, \Delta)$ with cobracket, induced from cocommutative coassociative coalgebra structure, defined by

$$\Delta(f) = (\partial_1^\circ \otimes \partial_2^\circ - \partial_2^\circ \otimes \partial_1^\circ)\mu^\circ(f) \ \text{ for } \ f \in \mathcal{A}^\circ, \tag{3.1}$$

where $\partial_1, \partial_2 \in \mathrm{Der}\mathcal{A}$ are two fixed derivations satisfying $\partial_1\partial_2 = \partial_2\partial_1$. Here and below, for any $\partial \in \mathrm{Der}(\mathcal{A})$, we denote $\partial^\circ = \partial^*|_{\mathcal{A}^\circ}$.

Another approach is as follows: Let $\mathcal{A}_\varphi = (\mathcal{A}, [\cdot,\cdot])$ be the Lie algebra defined in (1.2) (where $\varphi = [\cdot,\cdot]$). The Lie coalgebra induced from $\mathcal{A}_\varphi$ is $\mathcal{A}^\circ_\varphi = (\mathcal{A}^\circ_\varphi, \varphi^\circ)$, where the subspace $\mathcal{A}^\circ_\varphi$ of $\mathcal{A}^*$ is determined by Proposition 2.5 with cobracket defined by

$$\varphi^\circ(f) = (\mu(\partial_1 \otimes \partial_2 - \partial_2 \otimes \partial_1))^*(f) = (\partial_1 \otimes \partial_2 - \partial_2 \otimes \partial_1)^*\mu^*(f) \ \text{ for } \ f \in \mathcal{A}^\circ_\varphi. \tag{3.2}$$

Proposition 3.1. Let $(\mathcal{A}, \mu)$ be a commutative associative algebra with unit, and $\partial_1, \partial_2 \in \mathrm{Der}(\mathcal{A})$ are commutative. Then the Lie coalgebra $\mathcal{A}^\circ_\mu$ is a Lie subcoalgebra of $\mathcal{A}^\circ_\varphi$.

Proof. For $f \in \mathcal{A}^\circ_\mu$, we have $\varphi^\circ(f) = (\partial_1 \otimes \partial_2 - \partial_2 \otimes \partial_1)^*\mu^*(f) = (\partial_1 \otimes \partial_2 - \partial_2 \otimes \partial_1)^*\mu^\circ(f) = (\partial_1^\circ \otimes \partial_2^\circ - \partial_2^\circ\partial_1^\circ)\mu^\circ(f) = \Delta(f)$, where the last equality follows from (3.1). Thus, $\mathcal{A}^\circ_\mu$ is a Lie subcoalgebra of $\mathcal{A}^\circ_\varphi$. □

Theorem 3.2. Let $(\mathcal{A}, \mu)$ be a commutative associative algebra, and $(\mathcal{A}, \varphi)$ the Poisson Lie algebra defined in (1.2). If there exists $h \in \mathcal{A}$ such that the ideal I of $(\mathcal{A}, \varphi)$ generated by h has finite codimension, then $\mathcal{A}^\circ_\mu = \mathcal{A}^\circ_\varphi$. In particularly, if $\mathcal{A} = \mathbb{F}[x, y]$, then $\mathcal{A}^\circ_\mu = \mathcal{A}^\circ_\varphi$.

Proof. Denote $\cdot$ and $\star$ the actions of $(\mathcal{A}, \mu)$ and $(\mathcal{A}, \varphi)$ on $\mathcal{A}^*$ respectively, i.e., $(f \cdot a)(b) = f(\mu(a, b))$, and $(f \star a)(b) = f(\varphi(a, b))$ for $a, b \in \mathcal{A}$, $f \in \mathcal{A}^*$. From the relation $\varphi(a, bc) = \varphi(a, c)b + \varphi(a, b)c$ for $a, b, c \in \mathcal{A}$, we have

$$(f \star a) \cdot b - (f \cdot b) \star a = f \cdot \varphi(a, b), \ \forall\, a, b \in \mathcal{A}.$$

If $f \in \mathcal{A}^\circ_\varphi$, i.e. $f \star \mathcal{A}$ is finite dimensional, then $f \cdot \varphi(b, \mathcal{A})$ is finite dimensional. Thus if the ideal I has finite codimension, and $f \cdot I$ is finite dimensional, it follows that $f \cdot \mathcal{A}$ is finite dimensional. From Proposition 2.5, we have $f \in \mathcal{A}^\circ_\mu$. □

Remark 3.3. The difference between $\mathcal{A}^\circ_\mu$ and $\mathcal{A}^\circ_\varphi$ is that $\mathcal{A}^\circ_\mu$, as a Lie coalgebra, is induced from coassociative coalgebra $(\mathcal{A}^\circ, \mu^\circ)$, and $\mathcal{A}^\circ_\mu$ is determined by $(\mu^*)^{-1}(\mathcal{A}^* \otimes \mathcal{A}^*)$ as a vector subspace of $\mathcal{A}^*$ (a good subspace of the dual of $(\mathcal{A}, \mu)$), but $\mathcal{A}^\circ_\varphi$ is the dual of the Lie algebra $(\mathcal{A}, \varphi)$ (determined by Proposition 2.5).

4 Dual Lie bialgebras of Poisson type

Poisson algebras (cf. (1.1)) have important algebra structures, which have close relations with the Virasoro algebra and vertex operator (super)algebras (e.g., [1]). They can be also regarded as special cases of Lie algebras of Block type. Therefore, some attentions have been paid on them and some related Lie algebras (e.g., [4, 7, 19–25, 27, 28]). In this section, we consider the dual structures of Poisson type Lie bialgebras.

The following result can be found in [26].

Proposition 4.1. Let A, B be commutative associative algebras, regarding $A^* \otimes B^* \subset (A \otimes B)^*$, then $A^\circ \otimes B^\circ = (A \otimes B)^\circ$.

Recall from [17] that the dual space of $\mathbb{F}[x]$ can be identified with the space $\mathbb{F}[[\varepsilon]]$. From [14, 18], and Proposition 4.1, we have

Proposition 4.2. 1. Let $f = \sum_{i=0}^{\infty} f_i \varepsilon^i \in \mathbb{F}[[\varepsilon]]$ with $f_i \in \mathbb{F}$. Then

$$\begin{aligned} f \in \mathbb{F}[x]^\circ &\iff f_n = h_1 f_{n-1} + h_2 f_{n-2} + \cdots + h_r f_{n-r} \text{ for some } r \in \mathbb{N},\ h_i \in \mathbb{F} \text{ and all } n > r \\ &\iff f \in \left\{ \frac{g(\varepsilon)}{h(\varepsilon)} \,\middle|\, g(\varepsilon), h(\varepsilon) \in \mathbb{F}[\varepsilon],\ h(0) \neq 0 \right\}. \end{aligned}$$

2. Denote $\mathcal{A} = \mathbb{F}[x, y]$, the polynomial algebra on tow variables x, y. Then

$$\mathcal{A}^\circ = \mathbb{F}[x, y]^\circ \cong \mathbb{F}[x]^\circ \otimes \mathbb{F}[y]^\circ.$$

Let $(\mathfrak{g}, \varphi, \delta)$ be a Lie bialgebra. The map φ^* induces a map $\varphi^\circ := \varphi^*|_{\mathfrak{g}^\circ} : \mathfrak{g}^\circ \to \mathfrak{g}^\circ \otimes \mathfrak{g}^\circ$, making $(\mathfrak{g}^\circ, \varphi^\circ)$ to be a Lie coalgebra. By [15, Proposition 3], the map $\delta^* : \mathfrak{g}^* \otimes \mathfrak{g}^* \hookrightarrow (\mathfrak{g} \otimes \mathfrak{g})^* \xrightarrow{\delta^*} \mathfrak{g}^*$ induces a map $\delta^\circ := \delta^*_{\mathfrak{g}^\circ \otimes \mathfrak{g}^\circ} : \mathfrak{g}^\circ \otimes \mathfrak{g}^\circ \to \mathfrak{g}^\circ$, making $(\mathfrak{g}^\circ, \delta^\circ)$ to be a Lie algebra. Thus we obtain a Lie bialgebra $(\mathfrak{g}^\circ, \delta^\circ, \varphi^\circ)$, the dual Lie bialgebra of $(\mathfrak{g}, \varphi, \delta)$.

Now take $\mathcal{A} = \mathbb{F}[x, y]$. Let ε^i, η^i be duals of $x^i, y^i \in \mathcal{A}$ respectively, namely, $\langle \varepsilon^i \eta^j, x^k y^l \rangle = \varepsilon^i \eta^j (x^k y^l) = \delta_{i,k} \delta_{j,l}$ for $i, j, k, l \in \mathbb{Z}_+$. Any element $u \in \mathcal{A}^*$ can be written as $u = \sum_{i,j} u_{i,j} \varepsilon^i \eta^j$ (possibly an infinite sum). Let $g = \sum_{k,l} g_{k,l} x^k y^l \in \mathcal{A}$ (a finite sum). Then

$$\langle u, g \rangle = u(g) = \sum_{i,j,k,l} u_{i,j} g_{k,l} \langle \varepsilon^i \eta^j, x^k y^l \rangle = \sum_{i,j,k,l} u_{i,j} g_{k,l} \delta_{i,k} \delta_{j,l} \text{ (a finite sum)}. \tag{4.1}$$

Let $\partial_1 = \frac{\partial}{\partial x}$, $\partial_2 = \frac{\partial}{\partial y}$. Then we have the Poisson Lie algebra $(\mathcal{A}, \varphi)$ defined by (1.5). From Theorem 3.2, it is easy to check that $\mathcal{A}^\circ_\varphi = \mathcal{A}^\circ_\mu$.

Convention 4.3. (1) If an undefined notation appears in an expression, we treat it zero; for instance $\varepsilon^i \eta^j = 0$ if $i < 0$ or $j < 0$.

(2) When there is no confusion, we use $[\cdot, \cdot]$ to denote the bracket in $\mathfrak{g}$ or $\mathfrak{g}^\circ$, i.e., $[\cdot, \cdot] = \varphi$ or δ°. We also use Δ to denote the cobracket in $\mathfrak{g}$ or $\mathfrak{g}^\circ$, i.e., $\Delta = \delta$ or φ°.

Let $m, n \in \mathbb{Z}_+$, and take $a = x^m y^n$, $b = xy \in \mathcal{A}$. Then $[a, b] = (m - n)a$. Thus by [10], the triple $(\mathcal{A}, [\cdot, \cdot], \Delta_r)$ with $r = a \otimes b - b \otimes a$ is a coboundary triangular Lie bialgebra whose cobracket is defined by

$$\Delta_r(f) = f \cdot r = [f, a] \otimes b + a \otimes [f, b] - [f, b] \otimes a - b \otimes [f, a] \text{ for } f \in \mathcal{A}. \tag{4.2}$$

Theorem 4.4. Let $(\mathcal{A}, [\cdot, \cdot], \Delta_r)$ be the coboundary triangular Lie bialgebra defined above. The dual Lie bialgebra of $\mathcal{A}$ is $(\mathcal{A}^\circ, [\cdot, \cdot], \Delta)$, where $\mathcal{A}^\circ$ is described by Proposition 4.2(2) with cobracket Δ defined by

$$\Delta(\varepsilon^m \eta^n) = \sum_{k+s=m+1,\ l+t=n+1} (kt - ls) \varepsilon^k \eta^l \otimes \varepsilon^s \eta^t, \tag{4.3}$$

and bracket $[\cdot, \cdot]$ uniquely determined by the skew-symmetry and the following

$$[\varepsilon^i \eta^j, \varepsilon^s \eta^t] = \begin{cases} (m(t+1) - n(s+1)) \varepsilon^{s+1-m} \eta^{t+1-n} & \text{if } (i, j) = (1, 1),\ (s, t) \neq (1, 1), \\ (s - t) \varepsilon^s \eta^t & \text{if } (i, j) = (m, n) \neq (s, t) \neq (1, 1), \\ 0 & \text{otherwise.} \end{cases} \tag{4.4}$$

Proof. Assume $\mu^\circ(\varepsilon^m\eta^n)=\sum_{k,l,.s,t\in\mathbb{Z}_+}c_{k,l,s,t}\varepsilon^k\eta^l\otimes\varepsilon^s\eta^t$ for some $c_{k,l,s,t}\in\mathbb{F}$. Then

$$c_{i,j,p,q}=\mu^\circ(\varepsilon^m\eta^n)(x^iy^j\otimes x^py^q)=\langle\varepsilon^m\eta^n,\mu(x^iy^j\otimes x^py^q)\rangle=\langle\varepsilon^m\eta^n,x^{i+p}y^{j+q}\rangle=\delta_{m,i+p}\delta_{n,j+q}.$$

Thus, $\mu^\circ(\varepsilon^m\eta^n)=\sum_{k+s=m,\,l+t=n}\varepsilon^k\eta^l\otimes\varepsilon^s\eta^t$. Assume $\partial_1^\circ(\varepsilon^i\eta^j)=\sum_{s,t}c_{s,t}\varepsilon^s\eta^t$. Then

$$c_{k,l}=\partial_1^\circ(\varepsilon^i\eta^j)(x^ky^l)=\varepsilon^i\eta^j(\partial_1(x^ky^l))=k\delta_{i,k-1}\delta_{j,l}=(i+1)\delta_{i+1,k}\delta_{j,l},$$

i.e., $\partial_1^\circ(\varepsilon^i\eta^j)=(i+1)\varepsilon^{i+1}\eta^j$. Similarly, $\partial_2^\circ(\varepsilon^i\eta^j)=(j+1)\varepsilon^i\eta^{j+1}$. From (3.1), we obtain

$$\Delta(\varepsilon^m\eta^n)\;=\;(\partial_1^\circ\otimes\partial_2^\circ-\partial_2^\circ\otimes\partial_1^\circ)\mu^\circ(\varepsilon^m\eta^j)=\sum_{k+s=m+1,\,l+t=n+1}(kt-ls)\varepsilon^k\eta^l\otimes\varepsilon^s\eta^t.$$

Therefore, (4.3) holds. Next, we verify (4.4). We have

$$\begin{aligned}\langle[\varepsilon^i\eta^j,\varepsilon^s\eta^t],x^ky^l\rangle&=\langle\varepsilon^i\eta^j\otimes\varepsilon^s\eta^t,(kn-lm)x^{k+m-1}y^{l+n-1}\otimes xy+(k-l)x^my^n\otimes x^ky^l\\&\quad-(kn-lm)xy\otimes x^{k+m-1}y^{l+n-1}-(k-l)x^ky^l\otimes x^my^n\rangle\\&=\langle(n(i+1)-m(j+1))\delta_{s,1}\delta_{t,1}\varepsilon^{i+1-m}\eta^{j+1-n}+(s-t)\delta_{i,m}\delta_{j,n}\varepsilon^s\eta^t\\&\quad-(n(s+1)-m(t+1))\delta_{i,1}\delta_{j,1}\varepsilon^{s+1-m}\eta^{t+1-n}-(i-j)\delta_{s,m}\delta_{t,n}\varepsilon^i\eta^j,x^ky^l\rangle.\end{aligned}\tag{4.5}$$

If $(i,j)=(1,1)$ and $(s,t)\neq(1,1)$, then (4.5) gives (note that $(m,n)\neq(1,1)$)

$$[\varepsilon\eta,\varepsilon^s\eta^t]=\begin{cases}(m(t+1)-n(s+1))\varepsilon^{s+1-m}\eta^{t+1-n} & \text{if } s+1-m\geqslant0,\ t+1-m\geqslant0,\\ 0 & \text{otherwise.}\end{cases}$$

Thus we obtain the first case of (4.4) (cf. Convention 4.3 (1)). If $(i,j)=(m,n)\neq(s,t)\neq(1,1)$, then (4.5) gives $[\varepsilon^m\eta^n,\varepsilon^s\eta^t]=(s-t)\varepsilon^s\eta^t$, which is the second case of (4.4). It remains to verify the last case of (4.4). By (4.5), we have $[\varepsilon^i\eta^j,\varepsilon^s\eta^t]=0$ if $i,s\neq1,m$ or $j,t\neq1,n$. We discuss the situations in two subcases.

Subcase 1. Assume $i=1$ (thus $j\neq1$). Then (4.5) becomes

$$[\varepsilon\eta^j,\varepsilon^s\eta^t]=\big(2n-m(j+1)\big)\delta_{s,1}\delta_{t,1}\varepsilon^{2-m}\eta^{j+1-n}+(s-t)\delta_{1,m}\delta_{j,n}\varepsilon^s\eta^t-(1-j)\delta_{s,m}\delta_{t,n}\varepsilon\eta^j.\tag{4.6}$$

Note that $\varepsilon^{2-m}=0$ if $m>2$, in this case, (4.6) becomes $[\varepsilon\eta^j,\varepsilon^s\eta^t]=-(1-j)\delta_{s,m}\delta_{t,n}\varepsilon\eta^j$, and we have (4.4). Now assume $m=0$. Then (4.6) gives $[\varepsilon\eta^j,\varepsilon^s\eta^t]=2n\delta_{s,1}\delta_{t,1}\varepsilon^2\eta^{j+1-n}-(1-j)\delta_{s,0}\delta_{t,n}\varepsilon\eta^j$, and we see that the last case of (4.4) holds in this case. Next assume $m=1$. Then (4.6) becomes $[\varepsilon\eta^j,\varepsilon^s\eta^t]=(2n-(j+1))\delta_{s,1}\delta_{t,1}\varepsilon\eta^{j+1-n}+(s-t)\delta_{j,n}\varepsilon^s\eta^t-(1-j)\delta_{s,1}\delta_{t,n}\varepsilon\eta^j$. Hence the last case of (4.4) holds. Finally assume $m=2$. By (4.6), we have

$$[\varepsilon\eta^j,\varepsilon^s\eta^t]\;=\;(2n-2(j+1))\delta_{s,1}\delta_{t,1}\eta^{j+1-n}-(1-j)\delta_{s,2}\delta_{t,n}\varepsilon\eta^j,\tag{4.7}$$

and the last case of (4.4) holds again.

Subcase 2. Assume $i=m\neq1\neq s$. We have $[\varepsilon^m\eta^j,\varepsilon^s\eta^t]=(s-t)\delta_{j,n}\varepsilon^s\eta^t-(m-j)\delta_{s,m}\delta_{t,n}\varepsilon^m\eta^j$ by (4.5), i.e., $[\varepsilon^m\eta^j,\varepsilon^s\eta^t]=(s-t)\varepsilon^s\eta^t$ if $j=n$, or $(j-m)\varepsilon^m\eta^j$ if $(s,t)=(m,n)$, or 0 otherwise. This completes the proof of the theorem. □

Proposition 4.5. Let $f(x,y)=\sum_{i=0}^m\sum_{j=0}^n a_{i,j}x^iy^j\in\mathbb{F}[x,y]$ with $a_{m,n}\neq0$, and $k,l\in\mathbb{Z}_+$, $c\in\mathbb{F}$. Denote $\operatorname{Supp}f=\{(i,j)\in\mathbb{Z}_+^2\mid a_{ij}\neq0\}$. Then

$$[x^ky^l,f(x,y)]=cf(x,y)\neq0\iff k=l=1,\quad j-i=c,\ \forall\,(i,j)\in\operatorname{Supp}f,\tag{4.8}$$

$$[x^ky^l,f(x,y)]=0\iff kj-li=0,\ \forall\,(i,j)\in\operatorname{Supp}f.\tag{4.9}$$

In particular, if denote $r=A\otimes B-B\otimes A$, where either $A=xy$, $B=\sum_{i=0}^n a_ix^iy^{c+i}$ for some $c\in\mathbb{Z}_+$ and $a_i\in\mathbb{F}$, or $A=x^ky^l$, $B=f(x,y)=\sum_{i=0}^m\sum_{j=0}^n a_{i,j}x^iy^j$ such that $kj-li=0$ for all $(i,j)\in\operatorname{Supp}f$, then r is a solution of classical Yang-Baxter equation.

Proof. We have $[x^ky^l, f(x,y)] = \sum_{i,j} a_{i,j}(kj-li)x^{k+i-1}y^{l+j-1}$. By comparing the coefficients of the highest term (i.e., x^my^n) and x^iy^j for all i,j, one immediately obtains (4.8) and (4.9). $\square$

First consider $A = xy$, $B = \sum_{i=0}^n a_ix^iy^{i+m}$ with $a_n \neq 0$ and $m \in \mathbb{Z}_+\backslash\{0\}$. Then $[A,B] = mB \neq 0$. The triple $(\mathcal{A}, [\cdot,\cdot], \Delta_r)$ with $r = A\otimes B - B\otimes A$ is a coboundary triangular Lie bialgebra of Poisson type with bracket defined by (1.5) and cobracket defined by

$$\Delta_r(g) = [g,A]\otimes B + A\otimes[g,B] - B\otimes[g,A] - [g,B]\otimes A,\ \forall\, g\in\mathcal{A}.$$

The following is one of the main results of the present paper.

Theorem 4.6. Let $(\mathcal{A}, [\cdot,\cdot], \Delta_r)$ be the coboundary triangular Lie bialgebra defined as above. The dual Lie bialgebra of $(\mathcal{A}, [\cdot,\cdot], \Delta_r)$ is $(\mathcal{A}^\circ, [\cdot,\cdot], \Delta)$, where $\mathcal{A}^\circ$ is described by Proposition 4.2(2), with cobracket Δ defined by (4.3) and bracket uniquely determined by the skew-symmetry and the following.

(1) In case $m \neq 0$, we have (cf. Convention 4.3 (1)), for $(s,t) \neq (1,1)$ and $p \neq q$,

$$[\varepsilon^p\eta^p, \varepsilon^s\eta^t] = \delta_{p,1}\sum_{i=0}^n a_i(si+sm+m-ti)\varepsilon^{s-i+1}\eta^{t-i-m+1}, \tag{4.10}$$

$$[\varepsilon^p\eta^q, \varepsilon^s\eta^t] = (p-q)\sum_{i=0}^n a_i\delta_{s,i}\delta_{t,i+m}\varepsilon^p\eta^q - (s-t)\sum_{i=0}^n a_i\delta_{p,i}\delta_{q,i+m}\varepsilon^s\eta^t. \tag{4.11}$$

(2) In case $m = 0$, we have, for $(s,t) \neq (1,1)$,

$$[\varepsilon\eta, \varepsilon^s\eta^t] = \sum_{i=2}^n a_i(s-t)i\varepsilon^{s-i+1}\eta^{t-i+1}, \tag{4.12}$$

$$[\varepsilon^p\eta^p, \varepsilon^s\eta^t] = (t-s)a_p\varepsilon^s\eta^t \quad\text{if}\quad p\in\{0,2,3,\cdots,n\}, \tag{4.13}$$

$$[\varepsilon^p\eta^q, \varepsilon^s\eta^t] = 0 \quad\text{if}\quad p\neq q,\ s\neq t. \tag{4.14}$$

Proof. Denote $C = [x^ky^l, xy] = (k-l)x^ky^l$ and $D = [x^ky^l, B] = \sum_{i=0}^n a_i(ki+km-li)x^{i+k-1}y^{i+l+m-1}$. We have

$$\langle[\varepsilon^p\eta^q, \varepsilon^s\eta^t], x^ky^l\rangle = \langle\varepsilon^p\eta^q\otimes\varepsilon^s\eta^t, C\otimes B + A\otimes D - B\otimes C - D\otimes A\rangle = P^{p,q}_{s,t} - P^{s,t}_{p,q}, \tag{4.15}$$

where (regarding k,l as fixed)

$$\begin{aligned} P^{p,q}_{s,t} &= (k-l)\delta_{p,k}\delta_{q,l}\sum_{i=0}^n a_i\delta_{s,i}\delta_{t,i+m} + \delta_{p,1}\delta_{q,1}\sum_{i=0}^n a_i(ki+km-li)\delta_{s,i+k-1}\delta_{t,i+m+l-1} \\ &= (p-q)\delta_{p,k}\delta_{q,l}\sum_{i=0}^n a_i\delta_{s,i}\delta_{t,i+m} + \delta_{p,1}\delta_{q,1}\sum_{i=0}^n a_i(si+sm+m-ti)\delta_{s-i+1,k}\delta_{t-i-m+1,l} \\ &= \langle H^{p,q}_{s,t}, x^ky^l\rangle, \quad\text{and where,} \end{aligned}$$

$$H^{p,q}_{s,t} = (p-q)\sum_{i=0}^n a_i\delta_{s,i}\delta_{t,i+m}\varepsilon^p\eta^q + \delta_{p,1}\delta_{q,1}\sum_{i=0}^n a_i(si+sm+m-ti)\varepsilon^{s-i+1}\eta^{t-i-m+1}. \tag{4.16}$$

Thus

$$[\varepsilon^p\eta^q, \varepsilon^s\eta^t] = H^{p,q}_{s,t} - H^{s,t}_{p,q}. \tag{4.17}$$

Assume $(s,t)\neq(1,1)$. First suppose $m\neq 0$. Then (4.15)–(4.17) give $[\varepsilon^p\eta^p, \varepsilon^s\eta^t] = 0$ for $p = q \neq 1$, and $[\varepsilon\eta, \varepsilon^s\eta^t] = \sum_{i=0}^n a_i(si+sm+m-ti)\varepsilon^{s-i+1}\eta^{t-i-m+1}$ for $p = q = 1$. We have (4.10). If $p\neq q$, we have $[\varepsilon^p\eta^q, \varepsilon^s\eta^t] = (p-q)\sum_{i=0}^n a_i\delta_{s,i}\delta_{t,i+m}\varepsilon^p\eta^q - (s-t)\sum_{i=0}^n a_i\delta_{p,i}\delta_{q,i+m}\varepsilon^s\eta^t$, by (4.15)–(4.17), and we have (4.11). Now suppose $m = 0$. Then (4.16) becomes

$$H^{p,q}_{s,t} = (p-q)\sum_{i=0}^n a_i\delta_{s,i}\delta_{t,i}\varepsilon^p\eta^q + \delta_{p,1}\delta_{q,1}\sum_{i=0}^n a_i(s-t)i\varepsilon^{s-i+1}\eta^{t-i+1}. \tag{4.18}$$

We have $[\varepsilon\eta, \varepsilon^s\eta^t] = \sum_{i=2}^n a_i(s-t)i\varepsilon^{s-i+1}\eta^{t-i+1}$ for $(p,q) = (1,1)$, i.e., we have (4.12). If $p = q \in \{0,2,3,...,n\}$, it is easy to see from (4.18) and (4.17) that we have (4.13). Finally assume $p\neq q$. One can easily obtain (4.14). $\square$

Remark 4.7. Theorems 4.4 and 4.6 and the following theorem provide us some examples of the nontriviality of coboundary triangular Lie bialgebras, even in the case of very trivial solution $r = A\otimes B - B\otimes A$ of CYBE with $[A, B] = 0$.

Theorem 4.8. Let $(k, l) \in \mathbb{Z}_+^2$ be fixed and $A = x^k y^l, B = f(x, y) = \sum_{(i,j)\in S} a_{i,j} x^i y^j \in \mathcal{A}$ with $a_{i,j} \in \mathbb{F}$, where $S = \mathrm{Supp}\, f$ is some subset of $\mathbb{Z}_+^2$ such that $kj - li = 0$ for $(i, j) \in S$. Denote $r = A\otimes B - B\otimes A$. Then $(\mathcal{A}, [\cdot,\cdot], \Delta_r)$ is a coboundary triangular Lie bialgebra of Poisson type. The dual Lie bialgebra of $(\mathcal{A}, [\cdot,\cdot], \Delta_r)$ is $(\mathcal{A}^\circ, [\cdot,\cdot], \Delta)$ with cobracket Δ defined as in Theorem 4.4 and bracket uniquely determined by the skew-symmetry and the following.

(1) If $(s, t) \neq (k, l)$, then

$$[\varepsilon^k\eta^l, \varepsilon^s\eta^t] = \begin{cases} (l-k)a_{s,t}\varepsilon\eta + \sum\limits_{(i,j)\in S} a_{i,j}(j-i)\varepsilon^{s-i+1}\eta^{t-j+1} - (l-k)a_{k,l}\varepsilon^{s-k+1}\eta^{t-l+1}, \\ \sum\limits_{(i,j)\in S} a_{i,j}(j(s+1)-i(t+1))\varepsilon^{s-i+1}\eta^{t-j+1} - (l(s+1)-k(t+1))a_{k,l}\varepsilon^{s-k+1}\eta^{t-l+1}, \\ (l-k)a_{s,t}\varepsilon\eta + \sum\limits_{(i,j)\in S} a_{i,j}(j-i)\varepsilon^{s-i+1}\eta^{t-j+1}, \\ \sum\limits_{(i,j)\in S} a_{i,j}(j(s+1)-i(t+1))\varepsilon^{s-i+1}\eta^{t-j+1}, \end{cases} \tag{4.19}$$

according to the following four cases

(i) $(s, t), (k, l) \in S$, (ii) $(s, t) \notin S, (k, l) \in S$, (iii) $(s, t) \in S, (k, l) \notin S$ or (iv) $(s, t), (k, l) \notin S$.

(2) If $(p, q) \neq (k, l)$, $(s, t) \neq (k, l)$, then

$$[\varepsilon^p\eta^q, \varepsilon^s\eta^t] = \begin{cases} (l-k)a_{s,t}\varepsilon^{p-k+1}\eta^{q-l+1} - (l-k)a_{p,q}\varepsilon^{s-k+1}\eta^{t-l+1} & \text{if } (p,q),(s,t)\in S, \\ (l(p+1)-k(q+1))a_{s,t}\varepsilon^{p-k+1}\eta^{q-l+1} & \text{if } (p,q)\notin S, (s,t)\in S, \\ 0 & \text{if } (p,q)\notin S, (s,t)\notin S. \end{cases} \tag{4.20}$$

Proof. Since $[A, B] = 0$, the triple $(\mathcal{A}, [\cdot,\cdot], \Delta_r)$ is obviously a coboundary triangular Lie bialgebra. We only need to determine the bracket relations. Note that $\langle[\varepsilon^p\eta^q, \varepsilon^s\eta^t], x^{m'}y^{n'}\rangle$ equals

$$\begin{aligned} &\Big\langle \varepsilon^p\eta^q \otimes \varepsilon^s\eta^t, [x^{m'}y^{n'}, A]\otimes B + A\otimes[x^{m'}y^{n'}, B] - B\otimes[x^{m'}y^{n'}, A] - [x^{m'}y^{n'}, B]\otimes A\Big\rangle \\ &= (m'l - n'k)\delta_{p,m'+k-1}\delta_{q,n'+l-1}\sum_{(i,j)\in S} a_{i,j}\delta_{s,i}\delta_{t,j} + \delta_{p,k}\delta_{q,l}\sum_{(i,j)\in S} a_{i,j}(m'j - n'i)\delta_{s,i+m'-1}\delta_{t,j+n'-1} \\ &\quad - (m'l - n'k)\delta_{s,m'+k-1}\delta_{t,n'+l-1}\sum_{(i,j)\in S} a_{i,j}\delta_{p,i}\delta_{q,j} - \delta_{s,k}\delta_{t,l}\sum_{(i,j)\in S} a_{i,j}(m'j - n'i)\delta_{p,i+m'-1}\delta_{q,j+n'-1} \\ &= \langle H_{s,t}^{p,q} - H_{p,q}^{s,t}, x^{m'}y^{n'}\rangle, \end{aligned} \tag{4.21}$$

where (noting that $a_{s,t} = 0$ if $(s, t) \notin S$)

$$H_{s,t}^{p,q} = \big(l(p+1) - k(q+1)\big)a_{s,t}\varepsilon^{p-k+1}\eta^{q-l+1} + \delta_{p,k}\delta_{q,l}\sum_{(i,j)\in S} a_{i,j}\big(j(s+1) - i(t+1)\big)\varepsilon^{s-i+1}\eta^{t-j+1}.$$

If $(p, q) = (k, l)$ and $(s, t) \neq (k, l)$, then (4.21) gives (noting that $a_{k,l} = 0$ if $(k, l) \notin S$)

$$\begin{aligned} [\varepsilon^k\eta^l, \varepsilon^s\eta^t] &= (l-k)a_{s,t}\varepsilon\eta + \sum_{(i,j)\in S} a_{i,j}\big(j(s+1) - i(t+1)\big)\varepsilon^{s-i+1}\eta^{t-j+1} \\ &\quad - \big(l(s+1) - k(t+1)\big)a_{k,l}\varepsilon^{s-k+1}\eta^{t-l+1}. \end{aligned}$$

In particular, if $(s, t), (k, l) \in S$, then (using the fact that $kj - li = 0$ for $(i, j) \in S$)

$$[\varepsilon^k\eta^l, \varepsilon^s\eta^t] = (l-k)a_{s,t}\varepsilon\eta + \sum_{(i,j)\in S} a_{i,j}(j-i)\varepsilon^{s-i+1}\eta^{t-j+1} - (l-k)a_{k,l}\varepsilon^{s-k+1}\eta^{t-l+1},$$

which gives the first case of (4.19). If $(s,t)\notin S,(k,l)\in S$, then

$$[\varepsilon^k\eta^l,\varepsilon^s\eta^t]=\sum_{(i,j)\in S}a_{i,j}\big(j(s+1)-i(t+1)\big)\varepsilon^{s-i+1}\eta^{t-j+1}-\big(l(s+1)-k(t+1)\big)a_{k,l}\varepsilon^{s-k+1}\eta^{t-l+1},$$

which gives the second case of (4.19). If $(s,t)\in S,(k,l)\notin S$, then $[\varepsilon^k\eta^l,\varepsilon^s\eta^t]=(l-k)a_{s,t}\varepsilon\eta+\sum_{(i,j)\in S}a_{i,j}(j-i)\varepsilon^{s-i+1}\eta^{t-j+1}$, which gives the third case of (4.19). If $(s,t)\notin S,(k,l)\notin S$, then $[\varepsilon^k\eta^l,\varepsilon^s\eta^t]=\sum_{(i,j)\in S}a_{i,j}(j(s+1)-i(t+1))\varepsilon^{s-i+1}\eta^{t-j+1}$, which completes the proof of (4.19).

Now assume $(p,q)\neq(k,l),(s,t)\neq(k,l)$. Then (4.21) gives

$$[\varepsilon^p\eta^q,\varepsilon^s\eta^t]=\big(l(p+1)-k(q+1)\big)a_{s,t}\varepsilon^{p-k+1}\eta^{q-l+1}-\big(l(s+1)-k(t+1)\big)a_{p,q}\varepsilon^{s-k+1}\eta^{t-l+1}.$$

If $(p,q),(s,t)\in S$, then $[\varepsilon^p\eta^q,\varepsilon^s\eta^t]=(l-k)a_{s,t}\varepsilon^{p-k+1}\eta^{q-l+1}-(l-k)a_{p,q}\varepsilon^{s-k+1}\eta^{t-l+1}$, giving the first case of (4.20). If $(p,q)\notin S,(s,t)\in S$, then $[\varepsilon^p\eta^q,\varepsilon^s\eta^t]=\big(l(p+1)-k(q+1)\big)a_{s,t}\varepsilon^{p-k+1}\eta^{q-l+1}$, giving the second case of (4.20). In case $(p,q)\notin S,(s,t)\notin S$, we have $[\varepsilon^p\eta^q,\varepsilon^s\eta^t]=0$, which completes the proof of the theorem. □

In the final part of the paper, we will present an example of a dual Lie bialgebra which has different feature from the previous dual Lie bialgebras (Theorems 4.4, 4.6 and 4.8).

Denote $A=x(1+y)(2+y)$, $B=x^2(1+y)^3(2+y)\in\mathcal{A}$. Then (1.5) shows $[A,B]=B$. Take $r=A\otimes B-B\otimes A$. We obtain a coboundary triangular Lie bialgebra $(\mathcal{A},[\cdot,\cdot],\Delta)$ with bracket defined by (1.5), and cobracket defined by

$$\Delta(f)=f\cdot r=[f,A]\otimes B+A\otimes[f,B]-[f,B]\otimes A-B\otimes[f,A].$$

Theorem 4.9. Let $(\mathcal{A},[\cdot,\cdot],\Delta)$ be the Lie bialgebra defined above. The dual Lie bialgebra of $(\mathcal{A},[\cdot,\cdot],\Delta)$ is $(\mathcal{A}^\circ,[\cdot,\cdot],\Delta)$, where $\mathcal{A}^\circ$ and Δ are defined in Theorem 4.4, and the bracket is uniquely determined by the skew-symmetry and the following. First, denote

$$\begin{aligned}&c_0=2,\ c_1=3,\ c_2=1,\quad\text{and}\quad c_j=0\text{ for }j\neq0,1,2,\\&k_0=2,\ k_1=7,\ k_2=9,\ k_3=5,\ k_4=1,\quad\text{and}\quad k_t=0\text{ for }\ 4<t\in\mathbb{Z}_+.\end{aligned}$$

Then

$$\begin{aligned}[\varepsilon\eta^j,\varepsilon^s\eta^t]\ =\ &c_j\Big((4s-2t+2)\varepsilon^{s-1}\eta^{t-3}+(15s-10t+5)\varepsilon^{s-1}\eta^{t-2}\\&+18(s-t)\varepsilon^{s-1}\eta^{t-1}+(7s-14t-7)\varepsilon^{s-1}\eta^t+4(t+1)\varepsilon^{s-1}\eta^{t+1}\Big)\ \text{if } s\neq1,2,\end{aligned}\tag{4.22}$$

$$\begin{aligned}[\varepsilon\eta^j,\varepsilon\eta^t]\ =\ &c_j\Big((6-2t)\eta^{t-3}+(20-10t)\eta^{t-2}+18(1-t)\eta^{t-1}-14t\eta^t+4(t+1)\eta^{t+1}\Big)\\&-c_t\Big((6-2j)\eta^{j-3}+(20-10j)\eta^{j-2}+18(1-j)\eta^{j-1}-14j\eta^j+4(j+1)\eta^{j+1}\Big),\end{aligned}\tag{4.23}$$

$$\begin{aligned}[\varepsilon\eta^j,\varepsilon^2\eta^t]\ =\ &k_t\Big((3-j)\varepsilon\eta^{j-1}+3(1-j)\varepsilon\eta^j-2(j+1)\varepsilon\eta^{j+1}\Big)\\&+c_j\Big((10-2t)\varepsilon\eta^{t-3}\\&\quad+(35-10t)\varepsilon\eta^{t-2}+18(2-t)\varepsilon\eta^{t-1}+(7-14t)\varepsilon\eta^t+4(t+1)\varepsilon\eta^{t+1}\Big),\end{aligned}\tag{4.24}$$

$$[\varepsilon^2\eta^j,\varepsilon^s\eta^t]=-k_j\Big((2s-t+1)\varepsilon^s\eta^{t-1}+3(s-t)\varepsilon^s\eta^t-2(t+1)\varepsilon^s\eta^{t+1}\Big)\ \text{if } s\neq1,2,\tag{4.25}$$

$$\begin{aligned}[\varepsilon^2\eta^j,\varepsilon^2\eta^t]\ =\ &k_t\Big((5-j)\varepsilon^2\eta^{j-1}+3(2-j)\varepsilon^2\eta^j-2(j+1)\varepsilon^2\eta^{j+1}\Big)\\&-k_j\Big((5-t)\varepsilon^2\eta^{t-1}+3(2-t)\varepsilon^2\eta^t-2(t+1)\varepsilon^2\eta^{t+1}\Big),\end{aligned}\tag{4.26}$$

$$[\varepsilon^i\eta^j,\varepsilon^s\eta^t]=0\ \text{ if }\ \ i\neq1,2,s\neq1,2.\tag{4.27}$$

Proof. Denote $C=[x^my^n,A],D=[x^my^n,B]$. By (1.5), we have

$$\begin{aligned}&C=[x^my^n,x(y+1)(y+2)]=(2m-n)x^my^{n+1}+3(m-n)x^my^n-2nx^my^{n-1},\\&D=[x^my^n,x^2(y+1)^3(y+2)]\end{aligned}$$

$$= (4m-2n)x^{m+1}y^{n+3} + (15m-10n)x^{m+1}y^{n+2} + 18(m-n)x^{m+1}y^{n+1} + (7m-14n)x^{m+1}y^{n} - 4nx^{m+1}y^{n-1}.$$

For $i,j,s,t \in \mathbb{Z}_+$, note that $\langle[\varepsilon^i\eta^j,\varepsilon^s\eta^t],x^my^n\rangle$ is equal to

$$\begin{aligned}&\langle\varepsilon^i\eta^j\otimes\varepsilon^s\eta^t,[x^my^n,A]\otimes B+A\otimes[x^my^n,B]-[x^my^n,B]\otimes A-B\otimes[x^my^n,A]\rangle\\ &=\langle\varepsilon^i\eta^j\otimes\varepsilon^s\eta^t,C\otimes B+A\otimes D-D\otimes A-B\otimes C\rangle=P^{i,j}_{s,t}-P^{s,t}_{i,j},\end{aligned}\tag{4.28}$$

where $P^{i,j}_{s,t}=\langle\varepsilon^i\eta^j,C\rangle\langle\varepsilon^s\eta^t,B\rangle+\langle\varepsilon^i\eta^j,A\rangle\langle\varepsilon^s\eta^t,D\rangle$, which is equal to

$$\begin{aligned}&\Big((2m-n)\delta_{i,m}\delta_{j,n+1}+3(m-n)\delta_{i,m}\delta_{j,n}-2n\delta_{i,m}\delta_{j,n-1}\Big)\\ &\quad\times\Big(2\delta_{s,2}\delta_{t,0}+7\delta_{s,2}\delta_{t,1}+9\delta_{s,2}\delta_{t,2}+5\delta_{s,2}\delta_{t,3}+\delta_{s,2}\delta_{t,4}\Big)\\ &+\Big(2\delta_{i,1}\delta_{j,0}+3\delta_{i,1}\delta_{j,1}+\delta_{i,1}\delta_{j,2}\Big)\\ &\quad\times\Big((4m-2n)\delta_{s,m+1}\delta_{t,n+3}+(15m-10n)\delta_{s,m+1}\delta_{t,n+2}+18(m-n)\delta_{s,m+1}\delta_{t,n+1}\\ &\qquad+(7m-14n)\delta_{s,m+1}\delta_{t,n}-4n\delta_{s,m+1}\delta_{t,n-1}\Big)\\ =&\ \Big\langle\Big(2\delta_{s,2}\delta_{t,0}+7\delta_{s,2}\delta_{t,1}+9\delta_{s,2}\delta_{t,2}+5\delta_{s,2}\delta_{t,3}+\delta_{s,2}\delta_{t,4}\Big)\\ &\quad\times\Big((2i-j+1)\varepsilon^i\eta^{j-1}+3(i-j)\varepsilon^i\eta^j-2(j+1)\varepsilon^i\eta^{j+1}\Big)\\ &\ +\Big(2\delta_{i,1}\delta_{j,0}+3\delta_{i,1}\delta_{j,1}+\delta_{i,1}\delta_{j,2}\Big)\\ &\quad\times\Big((4s-2t+2)\varepsilon^{s-1}\eta^{t-3}+(15s-10t+5)\varepsilon^{s-1}\eta^{t-2}+18(s-t)\varepsilon^{s-1}\eta^{t-1}\\ &\qquad+(7s-14t-7)\varepsilon^{s-1}\eta^t+4(t+1)\varepsilon^{s-1}\eta^{t+1}\Big),x^my^n\Big\rangle.\end{aligned}$$

We obtain

$$[\varepsilon^i\eta^j,\varepsilon^s\eta^t]=H^{i,j}_{s,t}-H^{s,t}_{i,j},\tag{4.29}$$

where $H^{i,j}_{s,t}$ is equal to

$$\begin{aligned}&\Big(2\delta_{s,2}\delta_{t,0}+7\delta_{s,2}\delta_{t,1}+9\delta_{s,2}\delta_{t,2}+5\delta_{s,2}\delta_{t,3}+\delta_{s,2}\delta_{t,4}\Big)\Big((2i-j+1)\varepsilon^i\eta^{j-1}+3(i-j)\varepsilon^i\eta^j-2(j+1)\varepsilon^i\eta^{j+1}\Big)\\ &+\Big(2\delta_{i,1}\delta_{j,0}+3\delta_{i,1}\delta_{j,1}+\delta_{i,1}\delta_{j,2}\Big)\\ &\quad\times\Big((4s-2t+2)\varepsilon^{s-1}\eta^{t-3}+(15s-10t+5)\varepsilon^{s-1}\eta^{t-2}+18(s-t)\varepsilon^{s-1}\eta^{t-1}\\ &\qquad+(7s-14t-7)\varepsilon^{s-1}\eta^t+4(t+1)\varepsilon^{s-1}\eta^{t+1}\Big).\end{aligned}$$

If $i\neq 1,2$, $s\neq 1,2$, then (4.29) gives that $[\varepsilon^i\eta^j,\varepsilon^s\eta^t]=0$, which is (4.27). Assume $i=1$, $s\neq 1,2$. Then $[\varepsilon\eta^j,\varepsilon^s\eta^t]=H^{1,j}_{s,t}$, which gives (4.22). If $i=1,s=1$, then $[\varepsilon\eta^j,\varepsilon\eta^t]$ is equal to

$$\begin{aligned}&\Big(2\delta_{j,0}+3\delta_{j,1}+\delta_{j,2}\Big)\Big((6-2t)\eta^{t-3}+(20-10t)\eta^{t-2}+18(1-t)\eta^{t-1}-14t\eta^t+4(t+1)\eta^{t+1}\Big)\\ &-\Big(2\delta_{t,0}+3\delta_{t,1}+\delta_{t,2}\Big)\Big((6-2j)\eta^{j-3}+(20-10j)\eta^{j-2}+18(1-j)\eta^{j-1}-14j\eta^j+4(j+1)\eta^{j+1}\Big),\end{aligned}$$

which implies (4.23). Assume $i=1,s=2$. Then (4.29) implies that $[\varepsilon\eta^j,\varepsilon^2\eta^t]$ is equal to

$$\begin{aligned}&\Big(2\delta_{t,0}+7\delta_{t,1}+9\delta_{t,2}+5\delta_{t,3}+\delta_{t,4}\Big)\Big((3-j)\varepsilon\eta^{j-1}+3(1-j)\varepsilon\eta^j-2(j+1)\varepsilon\eta^{j+1}\Big)\\ &+\Big(2\delta_{j,0}+3\delta_{j,1}+\delta_{j,2}\Big)\Big((10-2t)\varepsilon\eta^{t-3}+(35-10t)\varepsilon\eta^{t-2}+18(2-t)\varepsilon\eta^{t-1}\\ &\qquad+(7-14t)\varepsilon\eta^t+4(t+1)\varepsilon\eta^{t+1}\Big),\end{aligned}$$

which gives (4.24). If $i=2,s\neq 1,2$, then (4.29) shows that $[\varepsilon^2\eta^j,\varepsilon^s\eta^t]$ is equal to

$$-(2\delta_{j,0}+7\delta_{j,1}+9\delta_{j,2}+5\delta_{j,3}+\delta_{j,4})\Big((2s-t+1)\varepsilon^s\eta^{t-1}+3(s-t)\varepsilon^s\eta^t-2(t+1)\varepsilon^s\eta^{t+1}\Big),$$

which is (4.25). Finally if $i = 2, s = 2$, we see from (4.29) that $[\varepsilon^2\eta^j, \varepsilon^2\eta^t]$ is equal to

$$\Big(2\delta_{t,0} + 7\delta_{t,1} + 9\delta_{t,2} + 5\delta_{t,3} + \delta_{t,4}\Big)\Big((5-j)\varepsilon^2\eta^{j-1} + 3(2-j)\varepsilon^2\eta^j - 2(j+1)\varepsilon^2\eta^{j+1}\Big)$$
$$-\Big(2\delta_{j,0} + 7\delta_{j,1} + 9\delta_{j,2} + 5\delta_{j,3} + \delta_{j,4}\Big)\Big((5-t)\varepsilon^2\eta^{t-1} + 3(2-t)\varepsilon^2\eta^t - 2(t+1)\varepsilon^2\eta^{t+1}\Big),$$

and we obtain (4.26). The proof of the theorem is completed. □

5 Conclusion remark

Theorems 4.4, 4.6, 4.8 and 4.9 provide five classes of infinite dimensional Lie Bialgebras $(\mathcal{A}^\circ, [\cdot,\cdot], \Delta)$ of Poisson type. As by-products, we obtain five new classes of infinite dimensional Lie algebras $(\mathcal{L}, [\cdot,\cdot])$ with the underlining space $\mathcal{L} = \mathbb{F}[\varepsilon, \eta]$ (the polynomial algebra on two variables ε, η) and brackets defined respectively by (4.4), (4.10)–(4.11), (4.12)–(4.14), (4.19)–(4.20) and (4.22)–(4.27). We close the paper by proposing the following questions: In which conditions will Lie bialgebras $(\mathcal{A}^\circ, [\cdot,\cdot], \Delta)$ defined in Theorems 4.4, 4.6, 4.8 and 4.9 be coboundary triangular? What kinds of structure and representation theories will these Lie algebras $(\mathcal{L}, [\cdot,\cdot])$ have?

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References

1 Ai C R, Han J Z, Regularity of vertex operator superalgebras, Sci China Math, 2014, 57: 1025–1032
2 Block R E, Commutative Hopf algebras, Lie coalgebras, and divided powers, J. Algebra, 1985, 96: 275–306
3 Block R E, Leroux P, Generalized dual coalgebras of algebras, with applications to cofree coalgebras, J. Pure Appl. Algebra, 1985,36: 15–21
4 Chen H J, Li J B, Left-symmetric algebra structures on the twisted Heisenberg-Virasoro algebra, Sci China Math, 2014, 57: 469–476
5 Diarra B, On the definition of the dual Lie coalgebra of a Lie algebra, Publ. Mat., 1995, 39: 349–354
6 Drinfel'd V, Quantum groups, Proceedings ICM (Berkeley 1986), Providence: Amer Math Soc, 1987, 789-820
7 Gao M, Gao Y, Su Y C, Irreducible Quasi-Finite Representations of a Block Type Lie Algebra, Comm. Algebra, 2014, 42: 511–527
8 Griffing G, The dual coalgebra of certain infinite-dimensional Lie algebras, Comm. Algebra, 2002, 30: 5715–5724
9 Majid S, Foundations of quantum group theory, Cambridge University Press 1995
10 Michaelis W, A class of infinite dimensional Lie bialgebras containing the Virasoro algebras, Adv. Math., 1994, 107: 365–392
11 Michaelis W, The dual Lie bialgebra of a Lie bialgebra, Modular interfaces (Riverside, CA, 1995), 81–93, AMS/IP Stud. Adv. Math., 4, Amer. Math. Soc., Providence, RI, 1997
12 Nichols W D, The structure of the dual Lie coalgebra of the Witt algebras, J. Pure Appl. Algebra, 1990, 68: 359–364
13 Nichols W D, On Lie and associative duals, J. Pure Appl. Algebra, 1993, 87: 313–320
14 Peterson B, Taft E J, The Hopf algebra of linearly recursive sequences, Aequationes Mathematicae, 1980, 20: 1–17, University Waterloo
15 Taft E J, Witt and Virasoro algebras as Lie bialgebras, J. Pure Appl. Algebra, 1993, 87: 301–312
16 Ng S H, Taft E J, Classification of the Lie bialgebra structures on the Witt and Virasoro algebras, J. Pure Appl. Algebra, 2000, 151: 67–88
17 Song G A, Su Y C, Lie Bialgebras of generalized Witt type, Science in China: Series A Mathematics, 2006, 49(4): 533–544
18 Song G A, Su Y C, Dual Lie Bialgebras of Witt and Virasoro Types (in Chinese), Sci Sin Math, 2013, 43: 1093-1102
19 Song G A, Su Y C, Wu Y Z, Quantization of generalized Virasoro-like algebras, Linear Algebra and it's Applications, 2008, 428: 2888-2899
20 Su Y C, Quasifinite representations of a Lie algebra of Block type, J. Algebra, 2004, 276: 117–128
21 Su Y C, Xia C G, Xu Y, Quasifinite representations of a class of Block type Lie algebras (q), J. Pure Appl. Algebra, 2012, 216: 923–934

22 Su Y C, Xia C G, Xu Y, Classification of quasifinite representations of a Lie algebra related to Block type, J. Algebra, 2013, 39: 71–78
23 Su Y C, Xu X P, Zhang H C, Derivation-simple algebras and the structures of Lie algebras of Witt type, J. Algebra, 2000, 233: 642–662
24 Su Y C, Xu Y, Yue X Q, Indecomposable modules of the intermediate series over $W(a,b)$, Sci China Math, 2014, 57: 275–291
25 Su Y C, Zhao K M, Generalized Virasoro and super-Virasoro algebras and modules of the intermediate series, J. Algebra, 2002, 252: 1–19
26 Sweedler M E, Hopf Algebras, W. A. Benjamin, Inc. New York, 1969
27 Xu X P, New generalized simple Lie algebras of Cartan type over a field with characteristic 0, J. Algebra, 2000, 224: 23–58
28 Wu Y Z, Song G A, Su Y C, Lie bialgebras of generalized Virasoro-like type, Acta Math. Sinica Engl. Ser., 2006, 22: 1915–1922
29 Xin B, Song G A, Su Y C, Hamiltonian type Lie bialgebras, Science in China Series A: Mathematics, 2007, 50: 1267–1279
30 van den Essen A, The sixtieth anniversary of the Jacobian conjecture: a new approach, Polynomial automorphisms and related topics, Ann. Polon. Math., 2001, 76: 77–87
31 Yu J T, Remarks on the Jacobian conjecture, J. Algebra, 1997, 88: 90–96